Fano varieties of small net complexity

arXiv 2610.27676

§1 Intro / $\mathbb{C}$

Def For a sm. Fano X,

$$i_X := \max\{m \in \mathbb{Z}_{>0} \mid -K_X = mH \text{ for some Cartier } H\}$$ Fano index

$$\ell_X := \max\{-K_X \cdot C \mid C \subset X : \text{rat. curve}\}$$: pseudoindex

Generalized 2003

Mukai Conj ('88) For a sm Fano X of dim n,

$$\rho_X (\underbrace{i_X}_{\ell_X} - 1) \le n, \quad \left[\text{"}=\text{"} \iff X \simeq (\mathbb{P}^{\overbrace{i_X-1}^{\ell_X}})^{\rho_X} \right]$$

OpenAI announced they proved this.

Def (Gongyo) X : normal proj. var. s.t. $-K_X$ is $\mathbb{Q}$-Cartier & nef

$$\tau_X := \max\left\{ \sum a_i \mid -K_X \equiv \sum_{i=1}^{k} a_i L_i,\ a_i \in \mathbb{Q}_{>0},\ L_i \not\equiv 0 \text{ nef Cartier} \right\}$$

↑ total index of X

$$c_X := \dim X + \rho_X - \tau_X$$: nef complexity of X

e.g. · For a sm Fano X s.t. $\rho_X = 1$, $\tau_X = i_X$

· For $X = \prod \mathbb{P}^{n_i}$, $n = \dim X$

$$-K_X = \sum_{i=1}^{\rho_X} (n_i+1) H_i$$

$$H_i \longleftrightarrow p_i^* \mathcal{O}_{\mathbb{P}^{n_i}}(1) \qquad p_i : \prod \mathbb{P}^{n_i} \longrightarrow \mathbb{P}^{n_i}$$

$$\tau_X = \sum_{i=1}^{\rho_X} (n_i+1) = n + \rho_X \qquad \therefore c_X = 0.$$

Mukai-type Conj (Gongyo '23) For a sm. Fano X,

$\cdot\ C_X \geq 0$, $\left[\text{"}=\text{"} \iff X \simeq \prod \mathbb{P}^{n_i}\right]$

Thm (Enwright-Filipazzi-Gongyo-Moraga-Svaldi-Wang-W '25)

For a klt Fano X, $C_X \geq 0$

$\text{"}=\text{"} \iff X \simeq \prod \mathbb{P}^{n_i}$

We proved more general thm for generalized klt log CY pairs.

We also asked which sm Fanos have $C_X \leq 1$.

& completed the classification in dim 3.

§2. Main Thm

Thm A For a klt Fano X, the following hold.

(i) $C_X \geq 0$ & $\left[\text{"}=\text{"} \iff X \simeq \prod \mathbb{P}^{n_i}\right]$ ← give another pf.

(ii) If $C_X < 1$, then $C_X = 0$.

(iii) If $C_X = 1$, then $X \simeq Y \times \prod \mathbb{P}^{d_i}$ ——— (*)

where Y is (a) a normal quadric $Q^d \subset \mathbb{P}^{d+1}$

(b) $Bl_{\mathbb{P}^l} \mathbb{P}^d$ $(d \geq 2,\ 0 \leq l \leq d-2)$

(c) a hypersurf. of bidegree $(1,1)$ in $\mathbb{P}^a \times \mathbb{P}^b$ $(a, b \geq 2)$

(iv) If $1 < C_X < 2$, X is a product as in (*), where Y is

(a) $\mathbb{P}(1,1,q^{d-1})$, where $d \geq 2$ & $q \geq 3$

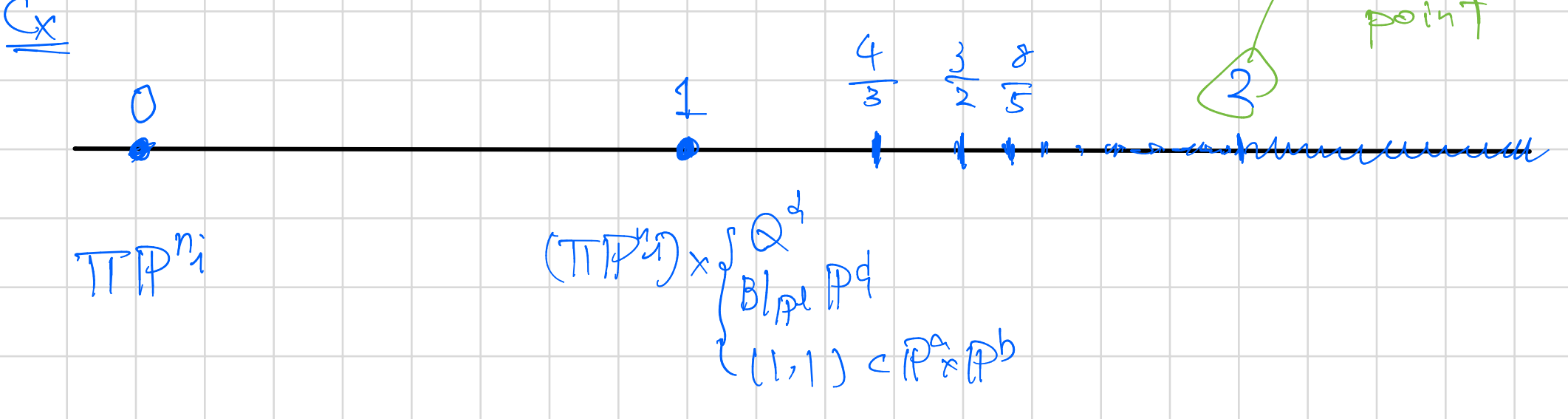
; in this case $C_X = 2 - \frac{2}{q}$.

(b) $\mathbb{P}(1,1,1,2^{d-2})$, where $d \geq 3$

in this case $C_X = \frac{3}{2}$

Thm B Let $\mathrm{Spec}_{klt}(c) := \{c_X \mid X : klt\ Fano\}$.

Then $\mathrm{Spec}_{klt}(c) = \{0\} \cup \{2 - \frac{2}{q} \mid q \in \mathbb{Z}_{\geq 2}\} \cup (\mathbb{Q} \cap [2, \infty))$



§3. A direct pf of $c_X \geq 0$

X : klt Fano of dim n.

Set $\Lambda := \mathrm{Pic}X / \equiv_{num} \subset N^1(X)_{\mathbb{R}}$

$C := \mathrm{Nef}(X) \subset N^1(X)_{\mathbb{R}}$: rational polyhedral cone

Gordan's lem $\Rightarrow$ $C \cap \Lambda$: finitely generated additive monoid.

$\Leftrightarrow$ has a unique minimal generating set.

$\Leftrightarrow$ "Hilbert basis"

For $\mathrm{Hilb}(C \cap \Lambda) = \{H_1, H_2, \cdots, H_N\}$,

define $B : \mathbb{R}^N \longrightarrow N^1(X)_{\mathbb{R}}$

Ψ $\qquad\qquad$ Ψ

$x = (x_\alpha) \longmapsto \sum_{\alpha=1}^{N} x_\alpha [H_\alpha]$

& put $F := B^{-1}([-K_X]) \cap \mathbb{R}^N_{\geq 0}$

$= \{ x \in \mathbb{R}^N_{\geq 0} \mid \sum_{\alpha=1}^{N} x_\alpha [H_\alpha] = [-K_X] \}$

$x = (x_1, \cdots, x_N)$

rational polyhedral cone.

$$l(x) := \sum_{\alpha=1}^{N} x_\alpha \quad \text{for } x \in F.$$

Prop The maximum of l on F equals τ_X and is attained at a rational vertex. Its positive coordinates give a decomp.

$$-K_X \equiv \sum_{i=1}^{s} a_i h_i \quad (a_i \in \mathbb{Q}_{>0}),\ \sum a_i = \tau_X \quad -(*)$$

"optimal vertex decomposition"

, where $[h_i]$'s are linearly indep.

In particular, $s \le \rho_X$.

Pf $F \neq \phi$.

Fix an ample cartier A

$$\mu := \min_{\alpha} (H_\alpha \cdot A^{n-1}) > 0.$$

For $x \in F$,

$$\mu\, l(x) = \min_{\alpha} (H_\alpha \cdot A^{n-1}) \sum_{\alpha=1}^{N} x_\alpha$$

$$\le \left(\sum x_\alpha H_\alpha\right) \cdot A^{n-1} = (-K_X) \cdot A^{n-1}$$

$$\rightsquigarrow \exists\, T = \max_{x \in F} l(x) \quad \& \quad \tau_X \le T$$

Consider $F_{max} := \{ x \in F \mid l(x) = T \}$: face of F.

Take a vertex v of F_{max}. v is also a vertex of F.

Since F is rat. polyhedral, v is rational.

$$\rightsquigarrow \tau_X = l(v) = T$$

Then $\{B(e_\alpha) \mid \nu_\alpha > 0\}$ is linearly indep.

$B(e_\alpha) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \alpha$ ← ∎

Take an optimal vertex decomp,

$$-K_X \equiv \sum_{i=1}^{S} a_i L_i \quad (a_i \in \mathbb{Q}_{>0}), \quad \sum a_i = \tau_X$$

$[L_1], \cdots, [L_S]$: lin. indep. $\Rightarrow S \le \rho_X$

$m_i := \lceil a_i \rceil - 1 \in \mathbb{Z}_{\ge 0}$ ⟸ $-K_X - \sum m_i L_i$

$= \sum (1 - \varepsilon_i) L_i$: ample

$0 \le \varepsilon_i := \lceil a_i \rceil - a_i = m_i + 1 - a_i < 1$

$M := \sum m_i, \quad e := \sum \varepsilon_i$

$$\tau_X = \sum_{i=1}^{S} a_i = \sum_{i=1}^{S} (m_i + 1 - \varepsilon_i) = M + S - e$$

$$P(t) := P(t_1, \cdots, t_S) := \chi(X, -\sum_{i=1}^{S} t_i L_i)$$

$D(t)$ ⟩ kv.

For $k = (k_1, \cdots, k_S) \in \mathbb{Z}^S$,

$$P(k) = \begin{cases} 0 & \text{for } (0 \le k_i \le m_i \ \forall i,\ k \ne \Phi) \\ 1 & \text{for } k = \Phi \end{cases}$$

For $m := (m_1, \cdots, m_S)$

$B_m := [0, m_1] \times [0, m_2] \times \cdots \times [0, m_S] \ni k \ne \Phi$

$\Rightarrow P(k) = 0.$

≪ vanishing box,

Consider $\Delta_i P(t) := P(t+\underbrace{e_i}_{\text{"}}) - P(t)$

, where $P(t) := \chi(X, \mathcal{O}_X(-D(t)))$ $\quad (0,\cdots,\underset{i}{1},\cdots 0)$

$D(t) := \sum t_i L_i$

Rmk $\deg(\Delta_i P) \le \deg P - 1$.

For $u = (u_1, \cdots, u_s) \in \mathbb{Z}^s_{\ge 0}$

$|u| := \sum u_i$, $\Delta^u := \Delta_1^{u_1}\Delta_2^{u_2}\cdots\Delta_s^{u_s}$

$$\Delta^u P(t) = \sum_{0\le k\le u} (-1)^{|u|-|k|} \binom{u}{k} P(t+k)$$

$$\binom{u}{k} = \prod_{i=1}^{s} \binom{u_i}{k_i}$$

Take $t=0$, $u=m$

Then $\Delta^m P(0) = \sum_{0\le k\le m} (-1)^{M-|k|} \binom{m}{k} P(k)$ ← Consider the vanishing box.

$$= (-1)^M P(0) = (-1)^M \neq 0$$

Since $\deg P(t) = n$, $\deg(\Delta_i P) \le \deg P - 1$

$$\deg(\Delta^m P) \le n - M$$

$\therefore M \le n$

$$c_X = n + \rho_X - \tau_X = n + \rho_X - (M + s - e)$$

$$= \underbrace{(n-M)}_{\ge 0} + \underbrace{(\rho_X - s)}_{\ge 0} + \underbrace{e}_{\ge 0} \ge 0$$

$c_X \le 1$